

KRONECKER PRODUCT GRAPHS AND COUNTING WALKS IN RESTRICTED LATTICES

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ABSTRACT. Formulas are derived for counting walks in the Kronecker product of graphs, and the associated spectral distributions are obtained by the Mellin convolution of probability distributions. Two-dimensional restricted lattices admitting the Kronecker product structure are listed, and their spectral distributions are calculated in terms of elliptic integrals.

1. INTRODUCTION

Counting walks in a graph is a basic and interesting problem. Let $G = (V, E)$ be a locally finite graph with adjacency matrix A . Then the matrix element $(A^m)_{xy}$ counts the number of m -step walks connecting x and y . For $x = y = o \in V$ this number is expressible in the integral:

$$(A^m)_{oo} = \int_{\mathbb{R}} x^m \mu(dx), \quad m = 0, 1, 2, \dots,$$

where μ is a probability distribution on $\mathbb{R} = (-\infty, +\infty)$, called the *spectral distribution* of A at a vertex o . Thus the number of walks may be studied from an analytic or probabilistic point of view. During the last fifteen years the quantum probability has been employed for the asymptotic analysis of graph spectra as well as the study of product structures in connection with several notions of independence, see e.g., a monograph [6].

This paper focuses on the notion of *Kronecker product* of graphs $G_1 \times_K G_2$, which is also called the *direct product* and is one of the most basic graph products, see the comprehensive monographs [5], [7]. It is well known that the spectral distribution of the Cartesian product of two graphs $G_1 \times_C G_2$ is obtained by the usual convolution of probability distributions defined by

$$\int_{\mathbb{R}} h(x) \mu_1 * \mu_2(dx) = \int_{\mathbb{R}} \int_{\mathbb{R}} h(x+y) \mu_1(dx) \mu_2(dy), \quad h \in C_{\text{bdd}}(\mathbb{R}).$$

The convolution $\mu_1 * \mu_2$ is known to be the distribution of the sum of two independent random variables $X_1 + X_2$. Quantum probability allows us to discuss variations of independence of non-commutative variables. The comb product of graphs is related to the monotone convolution, the star product to the Boolean convolution, and the free product to the free convolution, see e.g., [6], for further relevant results see [1].

The *Mellin convolution* in the original sense is the convolution product on the locally compact abelian group $(\mathbb{R}_{>0}, \cdot)$ with the Haar measure dx/x defined by

$$(1.1) \quad f \star g(x) = \int_0^\infty f(y) g\left(\frac{x}{y}\right) \frac{dy}{y} = \int_0^\infty f\left(\frac{x}{y}\right) g(y) \frac{dy}{y}$$

for $f, g \in L^1((0, \infty), dx/x)$, see e.g., [9]. Extending the above definition naturally to symmetric probability distributions on \mathbb{R} , we define the *Mellin convolution* $\mu_1 *_{\mathcal{M}} \mu_2$

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of two symmetric distributions μ_1 and μ_2 by

$$\int_{\mathbb{R}} h(x) \mu_1 *_M \mu_2(dx) = \int_{\mathbb{R}} \int_{\mathbb{R}} h(xy) \mu_1(dx) \mu_2(dy), \quad h \in C_{\text{bdd}}(\mathbb{R}).$$

Recall that a measure μ on \mathbb{R} is called symmetric if $\mu(-dx) = \mu(dx)$. The Kronecker product of graphs becomes a new member of the corresponding list of “product structures” of graphs and “convolution products” of probability distributions on \mathbb{R} .

This paper is organized as follows. In Section 2 we assemble basic notations and notions for counting walks in terms of the spectral distribution. In Section 3 we introduce the concept of Kronecker product of graphs and show some elementary properties with illustrations. The main result is stated in Theorem 3.7. In Section 4 two-dimensional integer lattices restricted to certain domains which admit the Kronecker product structure. We derive formulas for counting walks and show that the density functions of the spectral distributions are expressible in terms of elliptic integrals. Finally in Section 5 we discuss towards higher dimensional extension, where we find unexpectedly that the restricted integer lattice $\{x \geq y \geq z\}$ and the mixed product $(\mathbb{Z}_+ \times_K \mathbb{Z}_+) \times_C \mathbb{Z}_+$ are not isomorphic but have a common spectral distribution at the origin $(0, 0, 0)$.

2. COUNTING WALKS IN A GRAPH

A graph $G = (V, E)$ is a pair, where V is a non-empty set and E a subset of two-point subsets of V , i.e., $E \subset \{\{x, y\}; x, y \in V, x \neq y\}$. We deal with both finite and infinite graphs. If $\{x, y\} \in E$, we say that x and y are adjacent and write $x \sim y$. The degree of $x \in V$ is defined to be the number of vertices that are adjacent to x , and is denoted by $\deg x = \deg_G x$. A graph under consideration in this paper is always assumed to be locally finite, i.e., $\deg x < \infty$ for all vertices $x \in V$.

For $m = 1, 2, \dots$ an m -step walk from a vertex $x \in V$ to another $y \in V$ is an (ordered) sequence of vertices x_0, x_1, \dots, x_m such that

$$x = x_0 \sim x_1 \sim x_2 \sim \dots \sim x_{m-1} \sim x_m = y.$$

The number of such walks is interesting to study. The adjacency matrix of a graph $G = (V, E)$ is a matrix A indexed by $V \times V$ whose entries are defined by

$$(A)_{xy} = \begin{cases} 1, & \text{if } x \sim y; \\ 0, & \text{otherwise.} \end{cases}$$

By local finiteness the powers of A are well-defined and the matrix entry $(A^m)_{xy}$ counts the number of m -step walks connecting x and y . It is convenient to introduce the Hilbert space $\ell^2(V)$ of \mathbb{C} -valued square-summable functions on V with the inner product

$$\langle f, g \rangle = \sum_{x \in V} \overline{f(x)} g(x), \quad f, g \in \ell^2(V).$$

Let $\{\delta_x; x \in V\}$ be the canonical orthonormal basis of $\ell^2(V)$. Then we have

$$(A^m)_{xy} = \langle \delta_x, A^m \delta_y \rangle, \quad m = 0, 1, 2, \dots$$

We are particularly interested in counting the number of walks from a vertex $o \in V$ to itself, which is denoted by

$$W_m(o; G) = (A^m)_{oo}, \quad m = 0, 1, 2, \dots$$

We tacitly understand that $W_0(o; G) = 1$.

Theorem 2.1. *Let $G = (V, E)$ be a graph with a distinguished vertex $o \in V$. Then there exists a probability distribution μ on \mathbb{R} such that*

$$W_m(o; G) = (A^m)_{oo} = \langle \delta_o, A^m \delta_o \rangle = M_m(\mu), \quad m = 0, 1, 2, \dots,$$

where

$$M_m(\mu) = \int_{\mathbb{R}} x^m \mu(dx)$$

is the m -th moment of μ .

The proof is by the Hamburger theorem, see e.g., [6]. The probability distribution in Theorem 2.1 is called the *spectral distribution* of A in the vector state at $o \in V$. The spectral distribution is not uniquely determined in general due to the indeterminate moment problem, however, it is unique if the degrees of vertices are uniformly bounded, i.e., if $\sup\{\deg(x); x \in V\} < \infty$. If $W_{2m+1}(o; G) = (A^{2m+1})_{oo} = 0$ for all $m = 0, 1, 2, \dots$, the spectral distribution may be assumed to be symmetric.

3. KRONECKER PRODUCT OF GRAPHS

3.1. Definition and elementary properties. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two (finite or infinite) graphs with adjacency matrices $A^{(1)}$ and $A^{(2)}$, respectively. Let $V = V_1 \times V_2$ be the Cartesian product set and define a matrix A indexed by $V \times V$ by

$$(A)_{(x,y),(x',y')} = A_{xx'}^{(1)} A_{yy'}^{(2)}, \quad (x,y), (x',y') \in V.$$

Since A is a symmetric matrix whose diagonal entries are all zero and off-diagonal ones take values in $\{0, 1\}$, there exists a graph G on $V = V_1 \times V_2$ whose adjacency matrix is A , or equivalently, whose edge set is given by

$$E = \{(x,y), (x',y')\}; (A)_{(x,y),(x',y')} = 1\}.$$

The above graph G is called the *Kronecker product* of G_1 and G_2 , and is denoted by

$$G = G_1 \times_K G_2.$$

In other words, the Kronecker product of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a graph on $V_1 \times V_2$ with adjacency relation $(x,y) \sim_K (x',y') \iff x \sim x' \text{ and } y \sim y'$.

Remark 3.1. The term *Kronecker product* appears in [2] for instance, while there are many synonyms. The *direct product* is another common term used in [3], [5], [7] and so forth. In this paper we prefer to the former in order to avoid confusion with some terms in quantum probability.

Through the canonical unitary isomorphism $\ell^2(V_1 \times V_2) \cong \ell^2(V_1) \otimes \ell^2(V_2)$ given by $\delta_{(x,y)} \leftrightarrow \delta_x \otimes \delta_y$, the adjacency matrix A of $G_1 \times_K G_2$ is written as

$$(3.1) \quad A = A^{(1)} \otimes A^{(2)}.$$

In fact, by definition we have

$$(A)_{(x,y),(x',y')} = \langle \delta_{(x,y)}, A \delta_{(x',y')} \rangle = \langle \delta_x \otimes \delta_y, A(\delta_{x'} \otimes \delta_{y'}) \rangle$$

and

$$A_{xx'}^{(1)} A_{yy'}^{(2)} = \langle \delta_x, A^{(1)} \delta_{x'} \rangle \langle \delta_y, A^{(2)} \delta_{y'} \rangle = \langle \delta_x \otimes \delta_y, (A^{(1)} \otimes A^{(2)}) (\delta_{x'} \otimes \delta_{y'}) \rangle,$$

from which (3.1) follows.

We collect some elementary properties, of which the proofs are straightforward. For further relevant results, see the comprehensive monographs [5], [7].

Proposition 3.2. *For any graphs G_1, G_2, G_3 we have*

$$\begin{aligned} G_1 \times_K G_2 &\cong G_2 \times_K G_1, \\ (G_1 \times_K G_2) \times_K G_3 &\cong G_1 \times_K (G_2 \times_K G_3). \end{aligned}$$

Proposition 3.3. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs with $|V_1| \geq 2$ and $|V_2| \geq 2$. Then the Kronecker product $G_1 \times_K G_2$ has at most two connected components.*

Proposition 3.4. *Let P_1 be the graph consisting of a single vertex. Then for any graph $G = (V, E)$ the Kronecker product $P_1 \times_K G$ is a graph on V with no edges, i.e., an empty graph on V .*

The *Cartesian product* of two graphs G_1 and G_2 , denoted by $G_1 \times_C G_2$, is a graph on $V_1 \times V_2$ with adjacency matrix defined by

$$(A)_{(x,y),(x',y')} = A_{xx'}^{(1)} \delta_{yy'} + \delta_{xx'} A_{yy'}^{(2)},$$

or equivalently under the isomorphism $\ell^2(V_1 \times V_2) \cong \ell^2(V_1) \otimes \ell^2(V_2)$,

$$A = A^{(1)} \otimes I^{(2)} + I^{(1)} \otimes A^{(2)},$$

where $I^{(i)}$ is the identity matrix indexed by $V_i \times V_i$ for $i = 1, 2$.

The distance-2 graph of $G_1 \times_C G_2$ is a graph on $V_1 \times V_2$ with adjacency relation:

$$(3.2) \quad \begin{aligned} (x, y) \sim (x', y') &\iff \text{dis}_{G_1 \times_C G_2}((x, y), (x', y')) = 2 \\ &\iff \text{dis}_{G_1}(x, x') + \text{dis}_{G_2}(y, y') = 2. \end{aligned}$$

It is then easy to see that the Kronecker product $G_1 \times_K G_2$ is a subgraph of the distance-2 graph of $G_1 \times_C G_2$. However, $G_1 \times_K G_2$ is not necessarily an induced subgraph of the distance-2 graph of $G_1 \times_C G_2$.

3.2. Counting walks. The Kronecker product of graphs has a significant property from the viewpoint of counting walks.

Theorem 3.5. *Let $G_1 \times_K G_2$ be the Kronecker product of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. For $o_1 \in V_1$ and $o_2 \in V_2$ we have*

$$W_m((o_1, o_2); G_1 \times_K G_2) = W_m(o_1; G_1) W_m(o_2; G_2), \quad m = 0, 1, 2, \dots$$

Proof. Let $A^{(1)}$ and $A^{(2)}$ denote the adjacency matrices of G_1 and G_2 , respectively. Let A be the adjacency matrix of the Kronecker product $G_1 \times_K G_2$. Using the natural isomorphism $\ell^2(V_1 \times V_2) \cong \ell^2(V_1) \otimes \ell^2(V_2)$ and $A = A^{(1)} \otimes A^{(2)}$ as in (3.1) we calculate as follows:

$$\begin{aligned} W_m((o_1, o_2); G_1 \times_K G_2) &= \langle \delta_{(o_1, o_2)}, A^m \delta_{(o_1, o_2)} \rangle \\ &= \langle \delta_{o_1} \otimes \delta_{o_2}, (A^{(1)} \otimes A^{(2)})^m \delta_{o_1} \otimes \delta_{o_2} \rangle \\ &= \langle \delta_{o_1}, (A^{(1)})^m \delta_{o_1} \rangle \langle \delta_{o_2}, (A^{(2)})^m \delta_{o_2} \rangle \\ &= W_m(o_1; G_1) W_m(o_2; G_2), \end{aligned}$$

which completes the proof. \square

3.3. Mellin convolution of symmetric probability distribution on \mathbb{R} . We focus on symmetric probability distributions μ on \mathbb{R} having finite moments of all orders. Since $M_{2m+1}(\mu) = 0$ holds for all $m = 0, 1, 2, \dots$, we are mostly interested in the even moments. For such probability distributions μ and ν , there exists a probability distribution, denoted by $\mu *_M \nu$, uniquely specified by

$$\int_{\mathbb{R}} h(x) \mu *_M \nu(dx) = \int_{\mathbb{R}} \int_{\mathbb{R}} h(xy) \mu(dx) \nu(dy), \quad h \in C_{\text{bdd}}(\mathbb{R}).$$

We call $\mu *_M \nu$ the *Mellin convolution*. It is easily seen that $\mu *_M \nu$ is symmetric and has finite moments of all orders. In fact,

Proposition 3.6. *$M_m(\mu *_M \nu) = M_m(\mu) M_m(\nu)$ for all $m = 0, 1, 2, \dots$*

Combining Theorems 2.1, 3.5 and Proposition 3.6, we come to the following fundamental result.

Theorem 3.7. For $i = 1, 2$ let $G_i = (V_i, E_i)$ be a graph with a distinguished vertex o_i . Let μ_i be the spectral distribution of the adjacency matrix $A^{(i)}$ of G_i in the vector state at o_i . Assume that μ_i is symmetric, or equivalently that $W_{2m+1}(G_i, o_i) = 0$ for all $m = 0, 1, 2, \dots$ and $i = 1, 2$. Then we have

$$W_m((o_1, o_2); G_1 \times_K G_2) = M_m(\mu_1 *_M \mu_2), \quad m = 0, 1, 2, \dots$$

In other words, the spectral distribution of the Kronecker product $G_1 \times_K G_2$ in the vector state at (o_1, o_2) is the Mellin convolution of μ_1 and μ_2 .

The Mellin convolution is originally introduced on the basis of the locally compact abelian group $\mathbb{R}_{>0} = (0, \infty)$, see Introduction. In this connection we should note the following

Proposition 3.8. Let $f(x)$ and $g(x)$ be symmetric density functions on \mathbb{R} and consider the probability distributions $\mu(dx) = f(x)dx$ and $\nu(dx) = g(x)dx$. Then $\mu *_M \nu$ admits a symmetric density function $2f \star g(x)$, where $f \star g$ is the (original) Mellin convolution defined in (1.1).

Proof. By definition, for a symmetric function $h \in C_{\text{bdd}}(\mathbb{R})$ we have

$$\begin{aligned} \int_{\mathbb{R}} h(x) \mu *_M \nu(dx) &= \int_{\mathbb{R}} \int_{\mathbb{R}} h(xy) \mu(dx) \nu(dy) \\ &= 4 \int_0^\infty \int_0^\infty h(xy) f(x) g(y) dx dy \\ &= 4 \int_0^\infty g(y) dy \int_0^\infty h(x) f\left(\frac{x}{y}\right) \frac{dx}{y} \\ &= 2 \int_{\mathbb{R}} h(x) dx \int_0^\infty f\left(\frac{x}{y}\right) g(y) \frac{dy}{y}. \end{aligned}$$

Hence, $2f \star g(x)$ is the density function of $\mu *_M \nu$. \square

For the readers' convenience we make comparison with the Cartesian product. The classical convolution of two probability distributions μ and ν is a probability distribution, denoted by $\mu * \nu$, uniquely specified by

$$\int_{\mathbb{R}} h(x) \mu * \nu(dx) = \int_{\mathbb{R}} \int_{\mathbb{R}} h(x+y) \mu(dx) \nu(dy), \quad h \in C_{\text{bdd}}(\mathbb{R}).$$

By applying the binomial expansion we get the following.

Proposition 3.9. For $i = 1, 2$ let $G_i = (V_i, E_i)$ be a graph with a distinguished vertex o_i . Let μ_i be the spectral distribution of the adjacency matrix $A^{(i)}$ of G_i in the vector state at o_i . Then we have

$$\begin{aligned} W_m((o_1, o_2); G_1 \times_C G_2) &= \sum_{k=0}^m \binom{m}{k} W_k(o_1; G_1) W_{m-k}(o_2; G_2) \\ &= M_m(\mu_1 * \mu_2), \quad m = 0, 1, 2, \dots, \end{aligned}$$

where $\mu_1 * \mu_2$ is the (classical) convolution. In other words, the spectral distribution of the Cartesian product $G_1 \times_C G_2$ in the vector state at (o_1, o_2) is the convolution of μ_1 and μ_2 .

4. SUBGRAPHS OF 2-DIMENSIONAL LATTICE AS KRONECKER PRODUCTS

4.1. The Kronecker product $\mathbb{Z} \times_K \mathbb{Z}$. In order to avoid confusion we use the symbol \mathbb{Z}^2 just for the Cartesian product set. The Kronecker product $\mathbb{Z} \times_K \mathbb{Z}$ is by definition a graph on $\mathbb{Z}^2 = \{(u, v); u, v \in \mathbb{Z}\}$ with adjacency relation:

$$(4.1) \quad (u, v) \sim_K (u', v') \iff u' = u \pm 1 \quad \text{and} \quad v' = v \pm 1.$$

While, the so-called 2-dimensional integer lattice is a graph on \mathbb{Z}^2 with adjacency relation:

$$(x, y) \sim (x', y') \iff \begin{cases} x' = x \pm 1, \\ y' = y, \end{cases} \quad \text{or} \quad \begin{cases} x' = x, \\ y' = y \pm 1. \end{cases}$$

We see immediately from definition that $\mathbb{Z} \times_K \mathbb{Z}$ has two connected components, each of which is isomorphic to the 2-dimensional integer lattice $\mathbb{Z} \times_C \mathbb{Z}$. Denoting by $(\mathbb{Z} \times_K \mathbb{Z})^o$ the connected component of $\mathbb{Z} \times_K \mathbb{Z}$ containing $o = (0, 0)$, we claim the following

Theorem 4.1. $(\mathbb{Z} \times_K \mathbb{Z})^o \cong \mathbb{Z} \times_C \mathbb{Z}$, where the isomorphism preserves the origin.

Here we prepare a general result.

Proposition 4.2. For $i = 1, 2$ let $G_i = (V_i, E_i)$ be a graph and $H_i = (W_i, F_i)$ an induced subgraph of G_i . Then $H_1 \times_K H_2$ is an induced subgraph of $G_1 \times_K G_2$.

Proof. By definition the vertex set of $H_1 \times_K H_2$ is $W_1 \times W_2$. For two vertices $(x, y), (x', y') \in W_1 \times W_2$ we have $(x, y) \sim (x', y')$ in $H_1 \times_K H_2$ if and only if $x \sim x'$ in H_1 and $y \sim y'$ in H_2 by definition. Since H_1 and H_2 are respectively induced subgraphs of G_1 and G_2 , the last condition is equivalent to that $x \sim x'$ in G_1 and $y \sim y'$ in G_2 , hence to that $(x, y) \sim (x', y')$ in $G_1 \times_K G_2$. Consequently, $H_1 \times_K H_2$ is an induced subgraph of $G_1 \times_K G_2$ spanned by $W_1 \times W_2$. \square

4.2. Subgraphs of 2-dimensional integer lattice. For a subset $D \subset \mathbb{Z}^2$ let $L[D]$ denote the lattice restricted to D , i.e., the induced subgraph of $\mathbb{Z} \times_C \mathbb{Z}$ spanned by the vertices in D . We are particularly interested in restricted lattices which admit Kronecker product structure. Theorem 4.1 says that $\mathbb{Z} \times_C \mathbb{Z} = L[\mathbb{Z}^2]$ itself is isomorphic to the Kronecker product $(\mathbb{Z} \times_K \mathbb{Z})^o$.

Theorem 4.3. For $n \geq 2$ we have

$$L\{(x, y) \in \mathbb{Z}^2; x \geq y \geq x - (n - 1)\} \cong (P_n \times_K \mathbb{Z})^o,$$

where the right-hand side stands for the connected component of $P_n \times_K \mathbb{Z}$ containing $o = (0, 0)$, P_n being the path on $\{0, 1, \dots, n - 1\}$. Similarly,

$$L\{(x, y) \in \mathbb{Z}^2; x \geq y\} \cong (\mathbb{Z}_+ \times_K \mathbb{Z})^o.$$

Proof. The path P_n is naturally regarded as an induced subgraph of \mathbb{Z} spanned by $\{0, 1, 2, \dots, n - 1\}$. It then follows from Proposition 4.2 that $P_n \times_K \mathbb{Z}$ is an induced subgraph of $\mathbb{Z} \times_K \mathbb{Z}$. Therefore, $(P_n \times_K \mathbb{Z})^o$ is an induced subgraph of $(\mathbb{Z} \times_K \mathbb{Z})^o$. Then, in view of Figure 1, we see that $(P_n \times_K \mathbb{Z})^o$ is isomorphic to the induced subgraph of $\mathbb{Z} \times_C \mathbb{Z}$ spanned by $D = \{(x, y); x \geq y \geq x - (n - 1)\}$. The second assertion is proved similarly. \square

Theorem 4.4. For $k \geq 2$ and $l \geq 2$ we have

$$L\left\{(x, y) \in \mathbb{Z}^2; \begin{array}{l} 0 \leq x + y \leq k - 1, \\ 0 \leq x - y \leq l - 1 \end{array} \right\} \cong (P_k \times_K P_l)^o.$$

Moreover,

$$L\{(x, y) \in \mathbb{Z}^2; x \geq y \geq -x\} \cong (\mathbb{Z}_+ \times_K \mathbb{Z}_+)^o.$$

The proof is similar as above, see also Figure 1.

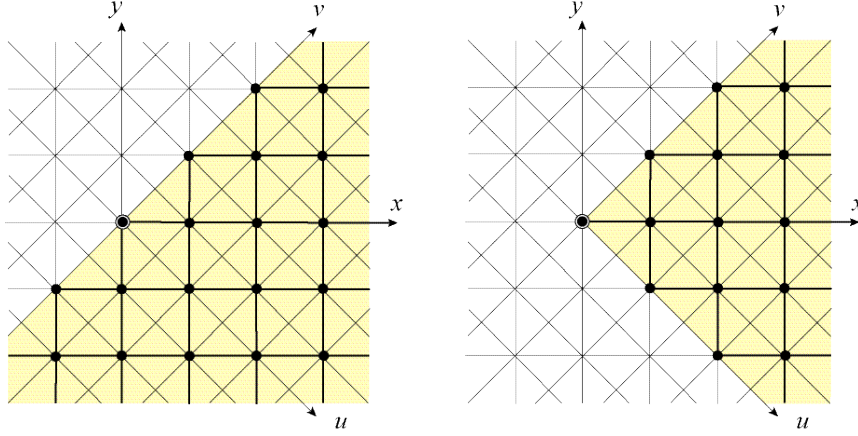


FIGURE 1. $(\mathbb{Z}_+ \times_K \mathbb{Z})^o \cong L(x \geq y)$ and $(\mathbb{Z}_+ \times_K \mathbb{Z}_+)^o \cong L(-x \leq y \leq x)$

4.3. Counting walks. The number of walks on one-dimensional integer lattice \mathbb{Z} from the origin 0 to itself is well known. We have

$$(4.2) \quad W_{2m}(0; \mathbb{Z}) = \binom{2m}{m}, \quad W_{2m+1}(0; \mathbb{Z}) = 0, \quad m = 0, 1, 2, \dots$$

A similar result for $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ is also well known. We have

$$(4.3) \quad W_{2m}(0; \mathbb{Z}_+) = C_m = \frac{1}{m+1} \binom{2m}{m}, \quad W_{2m+1}(0; \mathbb{Z}_+) = 0, \quad m = 0, 1, 2, \dots,$$

where C_m is the renowned Catalan number.

We start with typical restricted lattices.

Example 4.5. (1) For $L = L\{(x, y) \in \mathbb{Z}^2; x \geq y\}$ we have

$$W_{2m}((0, 0); L) = C_m \binom{2m}{m} = \frac{1}{m+1} \binom{2m}{m}^2, \quad m = 0, 1, 2, \dots,$$

and $W_{2m+1}((0, 0); L) = 0$. Indeed, by Theorem 4.3 we have $L \cong (\mathbb{Z}_+ \times_K \mathbb{Z})^o$, where the origin $(0, 0)$ in L corresponds to $o = (0, 0) \in \mathbb{Z}_+ \times_K \mathbb{Z}$. Hence

$$W_m((0, 0); L) = W_m((0, 0); \mathbb{Z}_+ \times_K \mathbb{Z}) = W_m(0; \mathbb{Z}_+) W_m(0; \mathbb{Z}),$$

where Theorem 3.5 is applied. Then the result follows from (4.2) and (4.3).

(2) For $L = L\{(x, y) \in \mathbb{Z}^2; x \geq y \geq -x\}$ we have

$$W_{2m}((0, 0); L) = C_m^2 = \frac{1}{(m+1)^2} \binom{2m}{m}^2, \quad m = 0, 1, 2, \dots,$$

and $W_{2m+1}((0, 0); L) = 0$. Indeed, we get the result from Theorem 4.4 along with a similar argument as in the previous example.

(3) For $L[\mathbb{Z}^2] = \mathbb{Z} \times_C \mathbb{Z}$ we have

$$(4.4) \quad W_{2m}((0, 0); \mathbb{Z} \times_C \mathbb{Z}) = \binom{2m}{m}^2, \quad m = 0, 1, 2, \dots$$

Indeed, from Theorem 4.1 we see that $L[\mathbb{Z}^2] = \mathbb{Z} \times_C \mathbb{Z} \cong (\mathbb{Z} \times_K \mathbb{Z})^o$. Then we obtain

$$\begin{aligned} W_{2m}((0, 0); \mathbb{Z} \times_C \mathbb{Z}) &= W_{2m}((0, 0); \mathbb{Z} \times_K \mathbb{Z}) \\ &= W_{2m}(0; \mathbb{Z}) W_{2m}(0; \mathbb{Z}) = \binom{2m}{m}^2, \end{aligned}$$

as desired. Formula (4.4) is derived in a different way. Applying Proposition 3.9 to the Cartesian product $\mathbb{Z} \times_C \mathbb{Z}$, we obtain

$$\begin{aligned} W_{2m}((0, 0); \mathbb{Z} \times_C \mathbb{Z}) &= \sum_{k=0}^m \binom{2m}{2k} W_{2k}(0; \mathbb{Z}) W_{2m-2k}(0; \mathbb{Z}) \\ &= \sum_{k=0}^m \binom{2m}{2k} \binom{2k}{k} \binom{2m-2k}{m-k}, \end{aligned}$$

where $W_{2m+1}(0; \mathbb{Z}) = 0$ is taken into account. By comparing with (4.4) we get the following interesting relation:

$$\sum_{k=0}^m \binom{2m}{2k} \binom{2k}{k} \binom{2m-2k}{m-k} = \binom{2m}{m}^2.$$

Of course, one may calculate the left-hand side directly by using the Vandermonde convolution formula for binomial coefficients to get the right-hand side.

Finally we record the case where $D \subset \mathbb{Z}^2$ is bounded in one or two directions, see Theorems 4.3 and 4.4.

Example 4.6. (1) For $L = L\{(x, y) \in \mathbb{Z}^2; x \geq y \geq x - (n - 1)\}$ with $n \geq 2$ we have

$$W_{2m}((0, 0); L) = W_{2m}(0; P_n) W_{2m}(0; \mathbb{Z}) = \binom{2m}{m} W_{2m}(0; P_n), \quad m = 0, 1, 2, \dots$$

(2) For $L = L\{(x, y) \in \mathbb{Z}^2; 0 \leq x + y \leq k - 1, 0 \leq x - y \leq l - 1\}$ with $k \geq 2$ and $l \geq 2$, we have

$$W_{2m}((0, 0); L) = W_{2m}(0; P_k) W_{2m}(0; P_l), \quad m = 0, 1, 2, \dots$$

Remark 4.7. A closed formula for $W_m(0; P_n)$ may be written down. Set

$$\lambda_k = 2 \cos \frac{k\pi}{n+1}, \quad k = 1, 2, \dots, n,$$

which are, in fact, obtained from zeroes of the Chebyshev polynomials of the second kind. We know that $\{\lambda_1, \dots, \lambda_n\}$ constitute the spectrum of P_n ([2, Section 1.4.4]). Then there exist real constants a_1, \dots, a_n such that

$$(4.5) \quad W_m(0; P_n) = \sum_{k=1}^n a_k \lambda_k^m, \quad m = 0, 1, 2, \dots$$

Then, (4.5) gives rise to a linear system $\mathbf{b} = \Lambda \mathbf{a}$. For $m \leq 2n$ we have

$$W_m(0; P_n) = W_m(0; \mathbb{Z}_+) = \begin{cases} C_{m/2}, & \text{if } m \text{ is even,} \\ 0, & \text{otherwise,} \end{cases}$$

and the Vandermonde matrix Λ is easily inverted, we obtain a_1, \dots, a_n uniquely from $\mathbf{a} = \Lambda^{-1} \mathbf{b}$. Here is a concrete example:

$$W_{2m}(0; P_4) = \frac{5 - \sqrt{5}}{10} \left(\frac{3 + \sqrt{5}}{2} \right)^m + \frac{5 + \sqrt{5}}{10} \left(\frac{3 - \sqrt{5}}{2} \right)^m$$

for $m = 0, 1, 2, \dots$, and, of course, $W_{2m+1}(0; P_4) = 0$.

4.4. Spectral distributions. We will describe spectral distributions corresponding to graphs with Kronecker product structures. We begin with their building blocks, namely, spectral distributions associated to \mathbb{Z} , \mathbb{Z}_+ and P_n .

The *arcsine distribution* with mean 0 and variance 2 is defined by the density function:

$$(4.6) \quad \alpha(x) = \frac{1}{\pi\sqrt{4-x^2}} 1_{(-2,2)}(x), \quad x \in \mathbb{R}.$$

The *semicircle distribution* with mean 0 and variance 1 is defined by the density function:

$$(4.7) \quad w(x) = \frac{1}{2\pi} \sqrt{4-x^2} 1_{[-2,2]}(x), \quad x \in \mathbb{R}.$$

By elementary calculus we have

$$(4.8) \quad M_{2m}(\alpha) = \int_{\mathbb{R}} x^{2m} \alpha(x) dx = \binom{2m}{m} = W_{2m}(0; \mathbb{Z}),$$

$$(4.9) \quad M_{2m}(w) = \int_{\mathbb{R}} x^{2m} w(x) dx = C_m = \frac{1}{m+1} \binom{2m}{m} = W_{2m}(0; \mathbb{Z}_+),$$

for $m = 0, 1, 2, \dots$. We see from Remark 4.7 that the spectral distribution π_n associated to P_n is given by

$$\pi_n = \sum_{k=1}^n a_k \delta_{\lambda_k},$$

where δ_x is the Dirac measure on the point $x \in \mathbb{R}$.

Now we move to the 2-dimensional cases associated to Cartesian and Kronecker products.

Example 4.8. For the Cartesian product $\mathbb{Z} \times_C \mathbb{Z}$ we have

$$\begin{aligned} W_m((0,0); \mathbb{Z} \times_C \mathbb{Z}) &= \sum_{k=0}^m \binom{m}{k} W_k(0; \mathbb{Z}) W_{m-k}(0; \mathbb{Z}) \\ &= \sum_{k=0}^m \binom{m}{k} M_k(\alpha) M_{m-k}(\alpha) = M_m(\alpha * \alpha). \end{aligned}$$

While, for the Kronecker product we have

$$\begin{aligned} W_m((0,0); \mathbb{Z} \times_K \mathbb{Z}) &= W_m(0; \mathbb{Z}) W_m(0; \mathbb{Z}) \\ &= M_m(\alpha) M_m(\alpha) = M_m(\alpha *_M \alpha). \end{aligned}$$

Since $\mathbb{Z} \times_C \mathbb{Z} \cong (\mathbb{Z} \times_K \mathbb{Z})^o$, we have

$$(4.10) \quad M_m(\alpha * \alpha) = M_m(\alpha *_M \alpha), \quad m = 0, 1, 2, \dots$$

Since $\alpha * \alpha$ (as well as $\alpha *_M \alpha$) has a compact support, (4.10) is sufficient to claim that $\alpha * \alpha = \alpha *_M \alpha$. By similar argument we obtain the spectral distributions for some restricted lattices. The following table summarizes the results.

Domain D	$W_{2m}(L[D], O)$	spectral distribution
\mathbb{Z}	$\binom{2m}{m}$	α
\mathbb{Z}_+	C_m	w
\mathbb{Z}^2	$\binom{2m}{m}^2$	$\alpha * \alpha = \alpha *_M \alpha$
$\{x \geq y\}$	$C_m \binom{2m}{m}$	$w *_M \alpha$
$\{x \geq y \geq -x\}$	C_m^2	$w *_M w$
$\{x \geq 0, y \geq 0\}$	(A)	$w * w$
$\{x \geq y \geq x - (n - 1)\}$	(B)	$\pi_n *_M \alpha$
$\left\{ \begin{array}{l} 0 \leq x + y \leq k - 1, \\ 0 \leq x - y \leq l - 1 \end{array} \right\}$	(C)	$\pi_k *_M \pi_l$

Concise formulas for (A)–(C) are not known, but we have

$$(A) = \sum_{k=0}^m \binom{2m}{2k} C_k C_{m-k}, \quad (B) = W_{2m}(0; P_n) \binom{2m}{m},$$

$$(C) = W_{2m}(0; P_k) W_{2m}(0; P_l).$$

4.5. Calculating density functions. In this section we investigate closed forms of density functions of the spectral distributions $\alpha * \alpha$, $w *_M \alpha$ and $w *_M w$.

Example 4.9. (1) It follows from Proposition 3.8 that the density function of $w *_M \alpha$ is given by $2w \star \alpha$. Since both $w(x)$ and $\alpha(x)$ are supported by the interval $[-2, 2]$, we see easily that $w \star \alpha(x) = 0$ for $x > 4$. Then, in terms of the explicit forms (4.6) and (4.7), we have:

$$(4.11) \quad \begin{aligned} w \star \alpha(x) &= \int_0^\infty w(y) \alpha\left(\frac{x}{y}\right) \frac{dy}{y} \\ &= \frac{1}{2\pi^2} \int_{x/2}^2 \sqrt{4-y^2} \frac{1}{\sqrt{4-(x/y)^2}} \frac{dy}{y} \\ &= \frac{1}{2\pi^2} \int_{x/2}^2 \sqrt{\frac{4-y^2}{4y^2-x^2}} dy, \quad 0 \leq x \leq 4. \end{aligned}$$

Here we need elliptic integrals and some relevant formulas [8]. The complete elliptic integrals of the first and second kinds are defined respectively by

$$\begin{aligned} K(k) &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}} = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2 x^2)}}, \\ E(k) &= \int_0^{\pi/2} \sqrt{1-k^2 \sin^2 \theta} d\theta = \int_0^1 \sqrt{\frac{1-k^2 x^2}{1-x^2}} dx, \end{aligned}$$

where $k^2 < 1$. Using the formula:

$$\int_b^a \sqrt{\frac{a^2-t^2}{t^2-b^2}} dt = a(K(k) - E(k)), \quad 0 < b < a, \quad k = \frac{\sqrt{a^2-b^2}}{a},$$

(4.11) becomes

$$w \star \alpha(x) = \frac{1}{2\pi^2} \{K(\xi(x)) - E(\xi(x))\},$$

where

$$\xi(x) = \sqrt{1 - \frac{x^2}{16}}.$$

Consequently, the density function of $w *_M \alpha$ is given by

$$\frac{1}{\pi^2} \{K(\xi(x)) - E(\xi(x))\} 1_{[-4,4]}(x), \quad x \in \mathbb{R}.$$

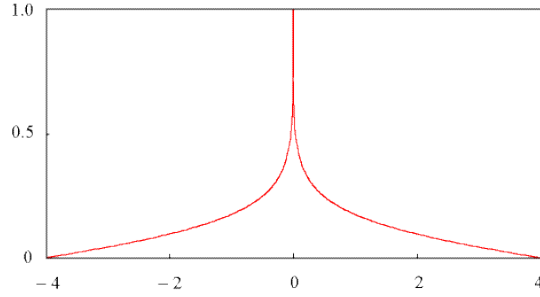


FIGURE 2. The density function of $w *_M \alpha$

(2) Similarly, the density function of $\alpha *_M \alpha = \alpha * \alpha$ is given by

$$\frac{1}{2\pi^2} K(\xi(x)) 1_{[-4,4]}(x), \quad x \in \mathbb{R},$$

and the density function of $w *_M w$ by

$$\frac{2}{\pi^2} \left\{ \left(1 + \frac{x^2}{16} \right) K(\xi(x)) - 2E(\xi(x)) \right\} 1_{[-4,4]}(x), \quad x \in \mathbb{R}.$$

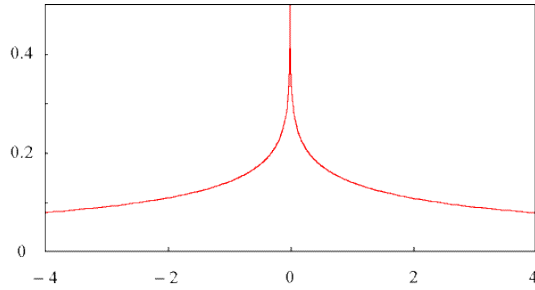


FIGURE 3. The density function of $\alpha *_M \alpha$

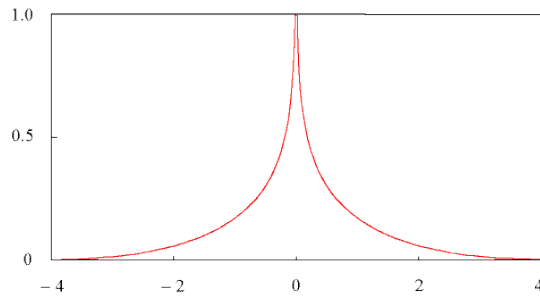


FIGURE 4. The density function of $w *_M w$

5. EXAMPLES IN HIGHER DIMENSION

In this section we focus on some higher dimensional examples. We begin with all possible combinations of products on \mathbb{Z}^3 , namely $\mathbb{Z} \times_K \mathbb{Z} \times_K \mathbb{Z}$, $(\mathbb{Z} \times_K \mathbb{Z}) \times_C \mathbb{Z}$, $(\mathbb{Z} \times_C \mathbb{Z}) \times_K \mathbb{Z}$ and $\mathbb{Z} \times_C \mathbb{Z} \times_C \mathbb{Z}$.

Example 5.1. (1) The Kronecker product $\mathbb{Z} \times_K \mathbb{Z} \times_K \mathbb{Z}$ has 4 connected components, which are mutually isomorphic. We have

$$W_{2m}((0,0,0); \mathbb{Z} \times_K \mathbb{Z} \times_K \mathbb{Z}) = \binom{2m}{m}^3, \quad m = 0, 1, 2, \dots$$

The connected component containing $O(0,0,0)$, as is illustrated in Figure 5, is the body-centered cubic lattice or a kind of octahedral honeycomb. For $(\mathbb{Z} \times_C \mathbb{Z}) \times_K \mathbb{Z}$ we have

$$((\mathbb{Z} \times_C \mathbb{Z}) \times_K \mathbb{Z})^o \cong ((\mathbb{Z} \times_K \mathbb{Z})^o \times_K \mathbb{Z})^o \cong (\mathbb{Z} \times_K \mathbb{Z} \times_K \mathbb{Z})^o.$$

Hence counting walks in $(\mathbb{Z} \times_C \mathbb{Z}) \times_K \mathbb{Z}$ is reduced to the previous one.

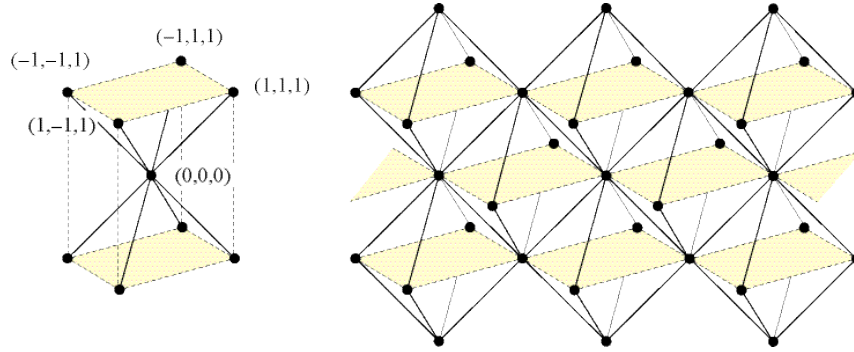


FIGURE 5. $(\mathbb{Z} \times_K \mathbb{Z} \times_K \mathbb{Z})^o$

(2) For other combinations of products of \mathbb{Z} we see that

$$((\mathbb{Z} \times_K \mathbb{Z}) \times_C \mathbb{Z})^o \cong (\mathbb{Z} \times_K \mathbb{Z})^o \times_C \mathbb{Z} \cong (\mathbb{Z} \times_C \mathbb{Z}) \times_C \mathbb{Z},$$

which is the usual 3-dimensional integer lattice. Hence

$$\begin{aligned} W_{2m}((0,0,0); (\mathbb{Z} \times_C \mathbb{Z}) \times_C \mathbb{Z}) &= W_{2m}((0,0,0); (\mathbb{Z} \times_K \mathbb{Z}) \times_C \mathbb{Z}) \\ &= \sum_{k=0}^m \binom{2m}{2k} \binom{2k}{k}^2 \binom{2m-2k}{m-k} = \sum_{k=0}^m \frac{(2m)!(2k)!}{(m-k)!2k!^4}. \end{aligned}$$

Of course the above result is well known, and our contribution here would be the derivation using the Kronecker product.

The last example is a very interesting case of products on \mathbb{Z}_+^3 , which is related to a restricted lattice in \mathbb{Z}^3 .

Example 5.2. The graph $(\mathbb{Z}_+ \times_K \mathbb{Z}_+) \times_C \mathbb{Z}_+$ has two connected components and we consider the connected component $((\mathbb{Z}_+ \times_K \mathbb{Z}_+) \times_C \mathbb{Z}_+)^o$ containing $O = (0,0,0)$.

Then we have

$$\begin{aligned}
(5.1) \quad W_{2m}((0, 0, 0); (\mathbb{Z}_+ \times_K \mathbb{Z}_+) \times_C \mathbb{Z}_+) \\
&= \sum_{k=0}^m \binom{2m}{2k} W_{2k}((0, 0); \mathbb{Z}_+ \times_K \mathbb{Z}_+) W_{2m-2k}(0; \mathbb{Z}_+) \\
&= \sum_{k=0}^m \binom{2m}{2k} C_k^2 C_{m-k} \\
&= \sum_{k=0}^m \frac{(2m)!(2k)!}{(m-k)!(m-k+1)!k!^2(k+1)!^2} .
\end{aligned}$$

It is remarkable that the last summation has been already obtained in [10] as the number of walks in the 3-dimensional restricted lattice $L\{x \geq y \geq z\} = \{(x, y, z) \in \mathbb{Z}^3 : x \geq y \geq z\}$, namely,

$$W_{2m}((0, 0, 0); (\mathbb{Z}_+ \times_K \mathbb{Z}_+) \times_C \mathbb{Z}_+) = W_{2m}((0, 0, 0); L\{x \geq y \geq z\}),$$

for all $m = 0, 1, 2, \dots$. It is, however, noted that $((\mathbb{Z}_+ \times_K \mathbb{Z}_+) \times_C \mathbb{Z}_+)^o$ and $L\{x \geq y \geq z\}$ are not isomorphic. For example, in the former graph there is a unique vertex with degree 2 (that is, $O = (0, 0, 0)$), while there are many vertices with degree 2 in the latter.

A similar phenomenon is observed also in the two-dimensional case.

Example 5.3. It follows by the usual reflection argument that

$$W_{2m}(1; \mathbb{Z}_+) = \binom{2m}{m} - \binom{2m}{m+2} = C_{m+1} .$$

On the other hand, it is known [4] that

$$W_{2m}((0, 0); \mathbb{Z} \times_C \mathbb{Z}_+) = \binom{2m}{m} \binom{2m+2}{m} - \binom{2m+2}{m+1} \binom{2m}{m-1} = C_m C_{m+1} .$$

Therefore,

$$W_{2m}((0, 0); \mathbb{Z} \times_C \mathbb{Z}_+) = W_{2m}((0, 1); \mathbb{Z}_+ \times_K \mathbb{Z}_+),$$

though two graphs $\mathbb{Z} \times_C \mathbb{Z}_+$ and $\mathbb{Z}_+ \times_K \mathbb{Z}_+$ are not isomorphic.

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